Simulation of Qubit Quantum Circuits via Pauli Propagation

quant-ph/1901.09070

The University of Texas at Austin
Quantum Information Center

Patrick Rall, Daniel Liang, Jeremy Cook, William Kretschmer
patrickjrall@gmail.com

Introduction

We study two novel **Pauli propagation** algorithms to estimate the outcomes of quantum circuits on qubits. They support any quantum circuit with noise, but are more efficient when the circuit is built from Clifford gates: Hadamard, Phase, and CNOT.

Bennink et al. [1] give a similar algorithm called **stabilizer propagation**, the only previously known noisy near-Clifford simulator for qubit circuits. Pashayan et al. [2] gave a protocol that works for qutrit circuits using the discrete Wigner function.

All these algorithms estimate the mean of some probability distribution via many samples. Pauli propagation takes linear time to sample, and never writes down a stabilizer state in the process. The number of samples can scale exponentially.

Runtime Analysis

Hoeffding inequality gives a condition to achieve accuracy ε with probability $1-\delta$ in terms of the maximum value of the distribution:

$$\#\text{ samples} \geq \frac{2}{\varepsilon^2}\cdot \ln\frac{2}{\delta}\cdot(\text{maximum})^2$$

The sampling algorithms output the product of many cost terms, one for each input state, quantum channel, and observable. When the cost is > 1 the range grows exponentially. Therefore, a component is **efficient** when its cost is less than 1.

Sampling Paulis

Let A be any hermitian matrix. Consider the completely dependent random variables $\hat{c}(A)$ and $\hat{\sigma}(A)$ below. They are an estimator: $\mathbb{E}(\hat{c}\cdot\hat{\sigma}) = A$.

$$\hat{\sigma}(A) = \sigma \text{ with prob. } \frac{|\mathrm{Tr}(\sigma A)|}{2^n\cdot\mathcal{D}(A)} \text{ for each Pauli } \sigma$$

$$\hat{c}(A) = \mathrm{sign}(\mathrm{Tr}(\hat{\sigma}A))\cdot\mathcal{D}(A)$$

The **stabilizer norm** $\mathcal{D}(A)$ is a constant that makes the above a PDF:

$$\mathcal{D}(A) = \frac{1}{2^n}\sum_{\text{Paulis }\sigma}|\mathrm{Tr}(\sigma A)|$$

When A is a tensor product of operators, each acting on a constant number of qubits, $\hat{c}$ and $\hat{\sigma}$ can be sampled from efficiently.

Sampling Stabilizer States

The **robustness of magic** $\mathcal{R}(\rho)$ gives another estimator $\mathbb{E}(\hat{d}\cdot\hat{\varphi}) = \rho$, via stab. states $\{|\varphi_i\rangle\langle\varphi_i|\}$:

$$\mathcal{R}(\rho) = \min_{\vec{q}}\sum_i |q_i| \text{ s.t. } \rho = \sum_i q_i\,|\varphi_i\rangle\langle\varphi_i|$$

$$\hat{\varphi}(\rho) = |\varphi_i\rangle\langle\varphi_i| \text{ and } \hat{d}(\rho) = \mathrm{sign}(q)\,\mathcal{R}(\rho) \text{ w.p. } \frac{|q_i|}{\mathcal{R}(\rho)}$$

Circuit: ρ (input state) $\rightarrow$ $\Lambda_1 \rightarrow \Lambda_2 \rightarrow \dots \rightarrow \Lambda_m$ (quantum channels) $\rightarrow$ E (observable)

Goal: estimate $\mathrm{Tr}[E\cdot\Lambda_m(\dots\Lambda_2(\Lambda_1(\rho))\dots)]$
`output` is an unbiased estimator

Schrödinger Propagation

1. Sample a $\hat{\sigma}$ and $\hat{c}$ for the input state ρ
   `operator = sample from` $\hat{\sigma}(\rho)$
   `output = sample from` $\hat{c}(\rho)$
   `maximum =` $\mathcal{D}(\rho)$
2. For each i, from 1 to m, sample a $\hat{\sigma}$ and $\hat{c}$ for Λ_i(`operator`)
   `operator = sample from` $\hat{\sigma}(\Lambda_i(\texttt{operator}))$
   `output *= sample from` $\hat{c}(\Lambda_i(\texttt{operator}))$
   `maximum *=` $\max_{\text{Paulis }\sigma}\mathcal{D}(\Lambda_i(\sigma))$
3. Take inner product with observable
   `output *=` $\mathrm{Tr}(\texttt{operator}\cdot E)$
   `maximum *=` $\max_{\text{Paulis }\sigma}\mathrm{Tr}(\sigma E)$
   `return output, maximum`

Heisenberg Propagation

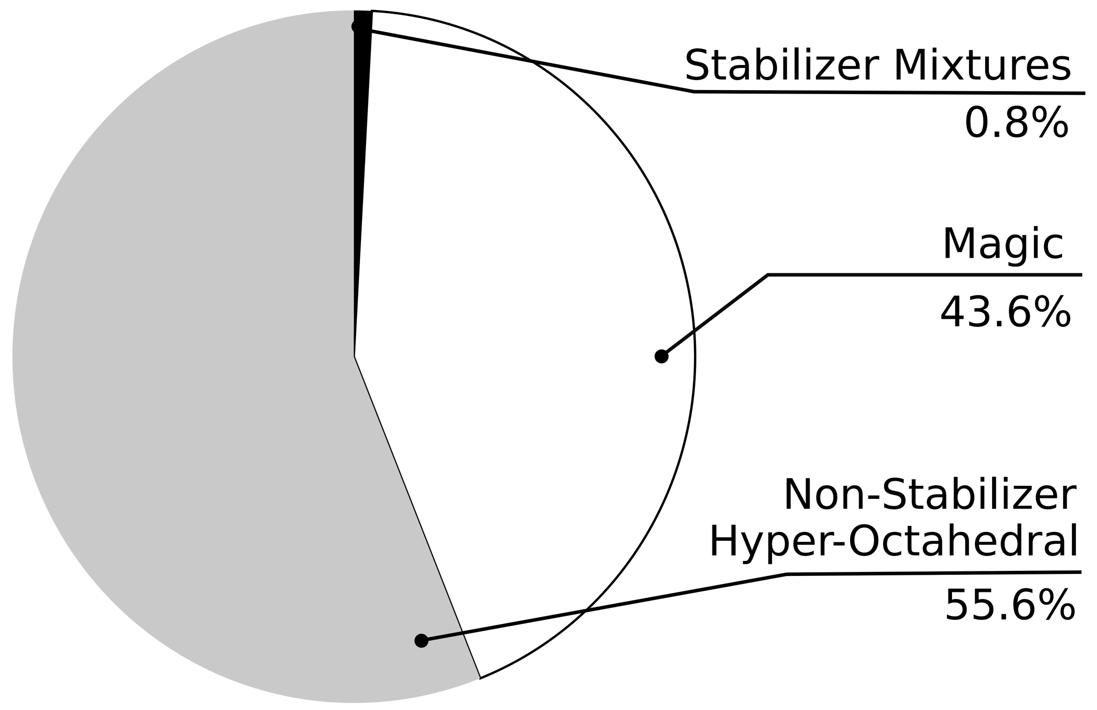
Channel adjoint $\Lambda^\dagger$ of Λ satisfies $\mathrm{Tr}(\Lambda(A)\cdot B) = \mathrm{Tr}(A\cdot\Lambda^\dagger(B))$

1. Sample a $\hat{\sigma}$ and $\hat{c}$ for the observable E
   `operator = sample from` $\hat{\sigma}(E)$
   `output = sample from` $\hat{c}(E)$
   `maximum =` $\mathcal{D}(E)$
2. For each i, from m to 1, sample a $\hat{\sigma}$ and $\hat{c}$ for $\Lambda_i^\dagger$(`operator`)
   `operator = sample from` $\hat{\sigma}(\Lambda_i^\dagger(\texttt{operator}))$
   `output *= sample from` $\hat{c}(\Lambda_i^\dagger(\texttt{operator}))$
   `maximum *=` $\max_{\text{Paulis }\sigma}\mathcal{D}(\Lambda_i^\dagger(\sigma))$
3. Take inner product with input ρ
   `output *=` $\mathrm{Tr}(\texttt{operator}\cdot\rho)$
   `maximum *= 1` $\left(= \max_{\text{Paulis }\sigma}\mathrm{Tr}(\sigma\rho)\right)$
   `return output, maximum`

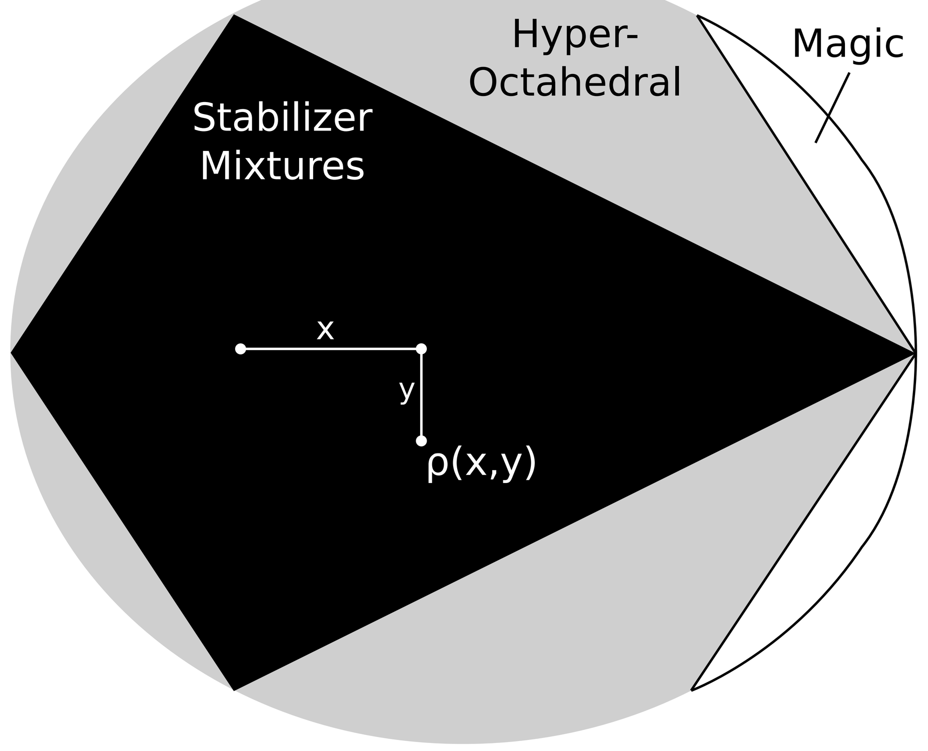
Previous Work: Stabilizer Propagation

1. Sample a $\hat{\varphi}$ and $\hat{d}$ for the input state ρ
   `operator = sample from` $\hat{\varphi}(\rho)$
   `output = sample from` $\hat{d}(\rho)$
   `maximum =` $\mathcal{R}(\rho)$
2. For each i, from 1 to m, sample a $\hat{\varphi}$ and $\hat{d}$ for Λ_i(`operator`)
   `operator = sample from` $\hat{\varphi}(\Lambda_i(\texttt{operator}))$
   `output *= sample from` $\hat{d}(\Lambda_i(\texttt{operator}))$
   `maximum *=` $\max_{\text{stab. states }|\varphi_i\rangle\langle\varphi_i|}\mathcal{R}(\Lambda_i(|\varphi_i\rangle\langle\varphi_i|))$
3. Take inner product with observable
   `output *=` $\mathrm{Tr}(\texttt{operator}\cdot E)$
   `maximum *=` $\max_{\text{stab. states }|\varphi_i\rangle\langle\varphi_i|}\mathrm{Tr}(|\varphi_i\rangle\langle\varphi_i|\,E)$
   `return output, maximum`

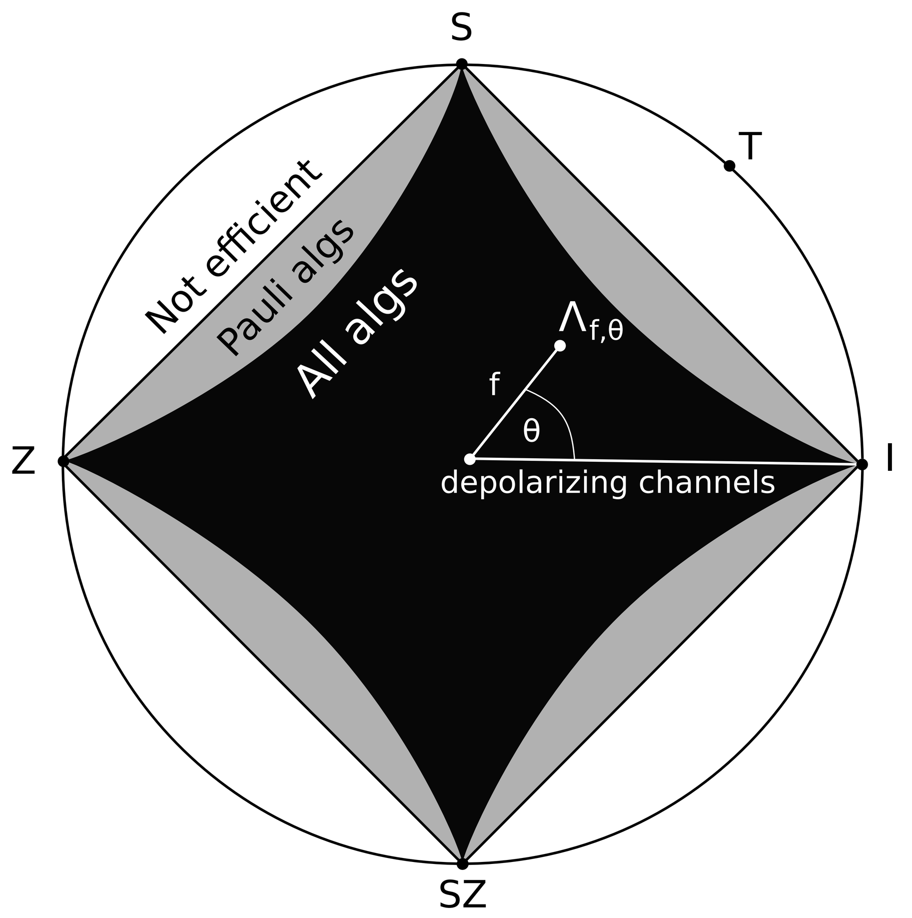
Two-Qubit States



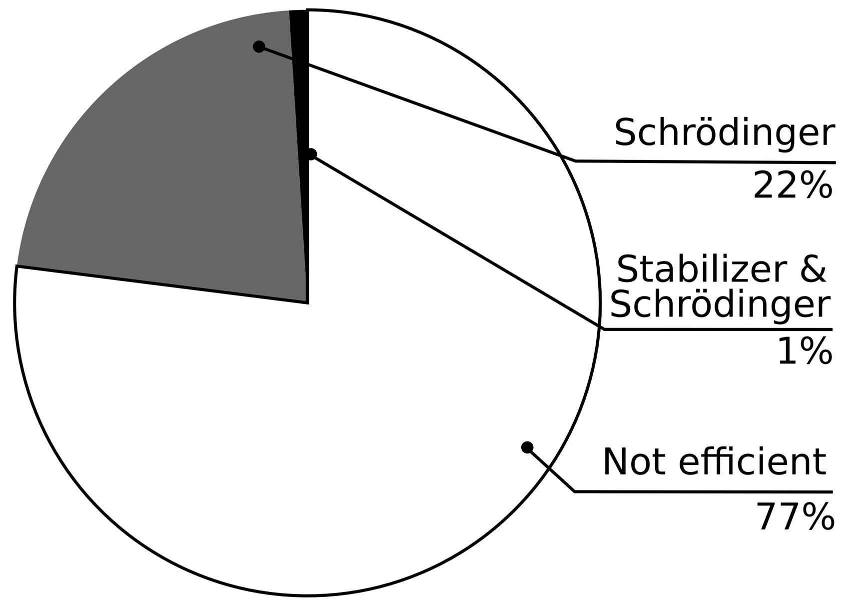
Two-Qubit Bloch Sphere Cross Section



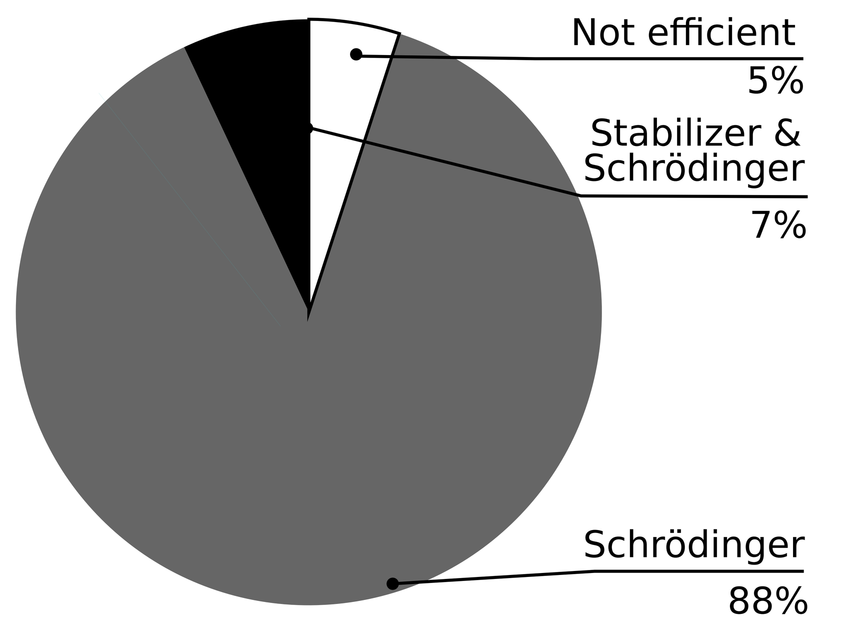
Depolarized Rotations



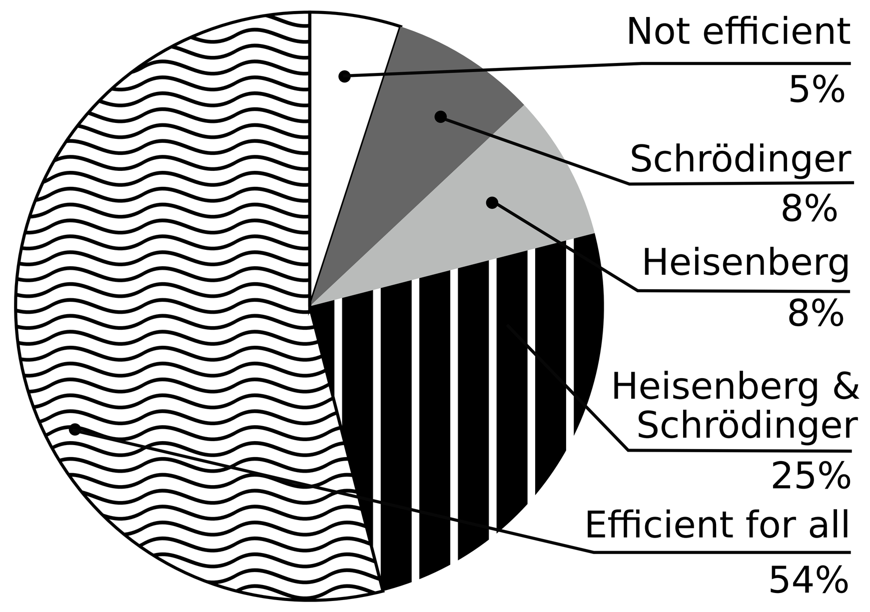
All Channels



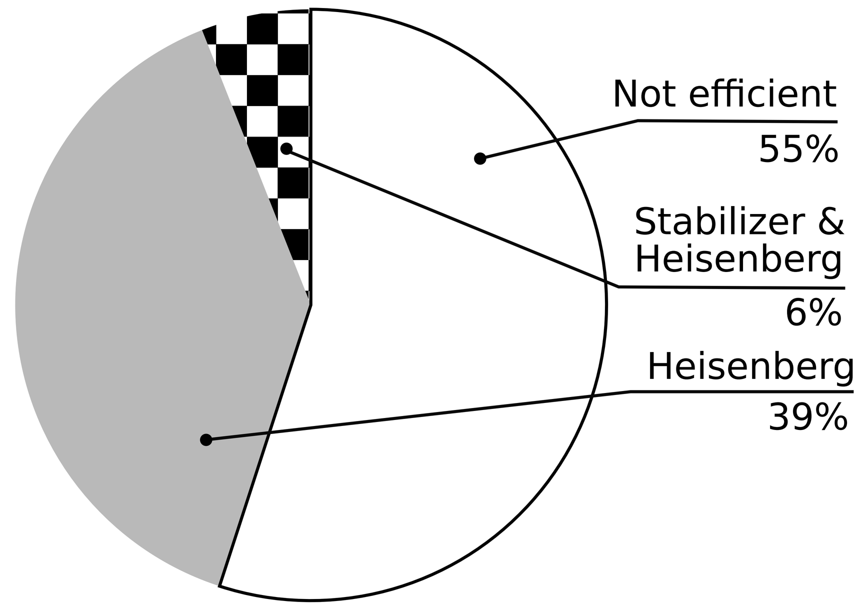
Unital Channels



Unital and TP Channels



Trace Preserving Channels



Efficient Input States

Stabilizer propagation can efficiently simulate input states that are **mixtures of stabilizer states**.

Schrödinger propagation can simulate larger class of **hyper-octahedral states**, defined by:

$$\mathcal{D}(\rho) \leq 1$$

According to the Hilbert-Schmidt measure, hyper-octahedral states are much more plentiful.

The runtime of Heisenberg propagation does not depend on the input state at all.

Efficient Channels

We exploit channel-state duality to map two-qubit states to qubit-to-qubit quantum channels. Pauli propagation methods consistently outperform stabilizer propagation for these channels.

Consider a channel $\Lambda_{f,\theta}$ which is the unitary $e^{-i\theta Z/2}$ composed with a depolarizing channel with fidelity f. Pauli propagation simulates more of these channels.

This work was supported by Scott Aaronson.

[1] quant-ph/1703.00111
[2] quant-ph/1503.07525